

XXV. *A new Method of investigating the Sums of infinite Series.*
 By the Rev. S. VINCE, A. M. of CAMBRIDGE, in a
Letter to Henry Maty, A. M. Secretary.

Read June 6, 1782.

S I R,

HAVING lately discovered some very easy methods of investigating the sums of certain infinite series, I have taken the liberty of requesting the favour of you to present them to the Royal Society. I have divided the subject into three parts: the first contains a new and general method of finding the sum of those series which DE MOIVRE has found in one or two particular cases; but whose method, although it be in appearance general, will, upon trial, be found to be absolutely impracticable. The second contains the summation of certain series, the last differences of whose numerators become equal to nothing. The third contains observations on a *correction* which is necessary in investigating the sums of certain series by collecting two terms into one, with its application to a variety of cases.

I am, &c.

Cambridge,
 May 3, 1782.

PART

P A R T I.

L E M. I.

Let r be any whole number, and then the fluent of $\frac{x}{1+x^r}$ can always be exhibited by circular arcs and logarithms; but when $x=1$, the fluent of the same fluxion will be expressed by the infinite series $1 - \frac{1}{r+1} + \frac{1}{2r+1} - \frac{1}{3r+1} + \&c.$ the sum of this series therefore can always be found by circular arcs and logarithms.

L E M. II.

To find the sum of the infinite series $\frac{a}{1 \cdot r+1} - \frac{a+b}{r+1 \cdot 2r+1} + \frac{a+2b}{2r+1 \cdot 3r+1} - \&c.$

Affume $1 - \frac{1}{r+1} + \frac{1}{2r+1} - \frac{1}{3r+1} + \&c. \dots = S$; therefore,

$$(A) \quad 1 - \frac{1}{1 \cdot r+1} + \frac{r+1}{r+1 \cdot 2r+1} - \frac{2r+1}{2r+1 \cdot 3r+1} + \&c. \dots = S.$$

In the first series, add together the 1st and 2d, the 2d and 3d, &c. &c. terms, and the resulting series will evidently be equal to twice that series *minus* the first term; therefore,

$$(B), \quad \frac{r}{1 \cdot r+1} - \frac{r}{r+1 \cdot 2r+1} + \frac{r}{2r+1 \cdot 3r+1} - \&c. \dots = 2S - 1.$$

$$\text{Now } \left(\frac{A}{r}\right) \frac{1}{r} - \frac{\frac{1}{r}}{1 \cdot r+1} + \frac{1 + \frac{1}{r}}{r+1 \cdot 2r+1} - \frac{2 + \frac{1}{r}}{2r+1 \cdot 3r+1} + \&c. \dots = \frac{S}{r},$$

$$\left. \begin{aligned} \text{or } & \frac{1}{r+1 \cdot 2r+1} - \frac{2}{2r+1 \cdot 3r+1} + \&c. \dots \\ & + \frac{1}{r} - \frac{\frac{1}{r}}{1 \cdot r+1} + \frac{\frac{1}{r}}{r+1 \cdot 2r+1} - \&c. \dots \end{aligned} \right\} = \frac{S}{r};$$

Now

Now the sum of the lower series, omitting the first term, is equal to $-B$ divided by r^2 , or $= -\frac{2S-1}{r^2}$; hence, by transposition, and, multiplying both sides by b , we shall have,

$$\frac{b}{r+1 \cdot 2r+1} - \frac{2b}{2r+1 \cdot 3r+1} + \&c. \dots = \frac{bS}{r} + \frac{2bS-r+1 \cdot b}{r^2};$$

also by multiplying B by $\frac{a}{r}$ we have

$$\frac{a}{1 \cdot r+1} - \frac{a}{r+1 \cdot 2r+1} + \frac{a}{2r+1 \cdot 3r+1} - \&c. \dots = \frac{2aS-a}{r};$$

subtract the last equation but one from the last, and we shall have

$$\frac{a}{1 \cdot r+1} - \frac{a+b}{r+1 \cdot 2r+1} + \frac{a+2b}{2r+1 \cdot 3r+1} - \&c. \dots = \frac{2ra-r+2 \cdot b \times S - ra+r+1 \cdot b}{r^2}$$

Cor. 1. Hence it appears, that the sum of this series can never be exhibited in finite terms, except $a : b$ as $r+2 : 2r$, in which case the sum is equal to $\frac{a}{r+2}$.

Hence, if $a=3, b=2$, then $r=1$; $\therefore \frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \&c. \dots = 1$;

if $a=1, b=4$, then $r=\frac{4}{3}$; $\therefore \frac{5}{3 \cdot 7} - \frac{9}{7 \cdot 11} + \frac{13}{11 \cdot 15} - \frac{17}{15 \cdot 19} + \&c. \dots = \frac{1}{6}$;

if $a=4, b=3$, then $r=\frac{6}{5}$; $\therefore \frac{4}{5 \cdot 11} - \frac{7}{11 \cdot 17} + \frac{10}{17 \cdot 23} - \frac{13}{23 \cdot 29} + \&c. \dots = \frac{1}{20}$.

Cor 2. Put $a=c-b$, and we shall have, after transposition,

$$\frac{c}{r+1 \cdot 2r+1} - \frac{c+b}{2r+1 \cdot 3r+1} + \&c. \dots = \frac{3r+2 \cdot b - 2rc \times S - 2r+1 \cdot b+rc}{r^2} + \frac{c-b}{r+1}$$

P R O P. I.

To find the sum of the infinite series $\frac{m}{1 \cdot r+1 \cdot 2r+1} +$

$$\frac{m+n}{2r+1 \cdot 3r+1 \cdot 4r+1} - \frac{m+2n}{4r+1 \cdot 5r+1 \cdot 6r+1} + \&c.$$

Every

Every series of this kind may be resolved into the following series $\frac{a}{1 \cdot r+1} - \frac{a+b}{r+1 \cdot 2r+1} + \frac{a+2b}{2r+1 \cdot 3r+1} - \frac{a+3b}{3r+1 \cdot 4r+1} + \&c.$ for if we reduce two terms of this series into one, it will become

$$\frac{2ar-b}{1 \cdot r+1 \cdot 2r+1} + \frac{2ra+2r-1 \cdot b}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{2ra+4r-1 \cdot b}{4r+1 \cdot 5r+1 \cdot 6r+1} + \&c.$$

where the denominators being the same as in the given series, and the numerators also in arithmetic progression, we have only to take a and b such quantities that the respective numerators may be also equal; assume, therefore, $2ra - b = m$, $2ra + 2r - 1 \cdot b = m + n$; therefore, $b = \frac{n}{2r}$, $a = \frac{2rm+n}{4r^2}$, which substituted for a and b in LEM. 2. gives

$$\frac{m}{1 \cdot r+1 \cdot 2r+1} + \frac{m+n}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{m+2n}{4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots$$

$$= \frac{2rm-r+1 \cdot n}{2r^3} \times S + \frac{2r+1 \cdot n-2rm}{4r^3}.$$

Let $r = 1$, and we have

$$\frac{m}{1 \cdot 2 \cdot 3} + \frac{m+n}{3 \cdot 4 \cdot 5} + \frac{m+2n}{5 \cdot 6 \cdot 7} + \&c. \dots = m - n \cdot S + \frac{3n-2m}{4}.$$

If $m = 1, n = 3, \frac{1}{1 \cdot 2 \cdot 3} + \frac{4}{3 \cdot 4 \cdot 5} + \frac{7}{5 \cdot 6 \cdot 7} + \&c. \dots = \frac{7}{4} - 2S;$

$m = 1, n = 0, \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \&c. \dots = S - \frac{1}{2}.$

Let $r = 2$, and the series becomes

$$\frac{m}{1 \cdot 3 \cdot 5} + \frac{m+n}{5 \cdot 7 \cdot 9} + \frac{m+2n}{9 \cdot 11 \cdot 13} + \&c. \dots = \frac{4m-3n}{16} \times S + \frac{5n-4m}{32}.$$

If $m = 1, n = 1, \frac{1}{1 \cdot 3 \cdot 5} + \frac{2}{5 \cdot 7 \cdot 9} + \frac{3}{9 \cdot 11 \cdot 13} + \&c. \dots = \frac{S}{16} + \frac{1}{32};$

$m = 1, n = 0, \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{5 \cdot 7 \cdot 9} + \frac{1}{9 \cdot 11 \cdot 13} + \&c. \dots = \frac{S}{4} - \frac{1}{8}.$

Let $r = 5$, and we shall have

$$\frac{m}{1 \cdot 6 \cdot 11} + \frac{m+n}{11 \cdot 16 \cdot 21} + \frac{m+2n}{21 \cdot 26 \cdot 31} + \&c. \dots = \frac{5m-3n}{125} \times S + \frac{11n-10m}{500}.$$

If $m=1, n=1, \frac{1}{1 \cdot 6 \cdot 11} + \frac{2}{11 \cdot 16 \cdot 21} + \frac{3}{21 \cdot 26 \cdot 31} + \&c. \dots = \frac{2}{125} \times S + \frac{1}{500}$;

$m=1, n=0, \frac{1}{1 \cdot 6 \cdot 11} + \frac{1}{11 \cdot 16 \cdot 21} + \frac{1}{21 \cdot 26 \cdot 31} + \&c. \dots = \frac{S}{25} - \frac{1}{50}$.

Cor. If $2r : r + 1 :: n : m$, the sum of the series can be accurately found, and will be equal to $\frac{m}{2r \cdot r + 1}$. Let therefore $m = r + 1$, and then $n = 2r$, consequently

$$\frac{1}{1 \cdot 2r + 1} + \frac{1}{2r + 1 \cdot 4r + 1} + \frac{1}{4r + 1 \cdot 6r + 1} + \&c. \dots = \frac{1}{2r};$$

which is also known from other principles.

P R O P. II.

To find the sum of the infinite series $\frac{m}{r + 1 \cdot 2r + 1 \cdot 3r + 1} + \frac{m+n}{3r + 1 \cdot 4r + 1 \cdot 5r + 1} + \frac{m+2n}{5r + 1 \cdot 6r + 1 \cdot 7r + 1} + \&c.$

This series resolves itself into

$$\frac{c}{r + 1 \cdot 2r + 1} - \frac{c+b}{2r + 1 \cdot 3r + 1} + \frac{c+2b}{3r + 1 \cdot 4r + 1} - \&c.;$$

for by reduction, as before, it becomes

$$\frac{2cr - r + 1 \cdot b}{r + 1 \cdot 2r + 1 \cdot 3r + 1} + \frac{2cr + r - 1 \cdot b}{3r + 1 \cdot 4r + 1 \cdot 5r + 1} + \frac{2cr + 3r - 1 \cdot b}{5r + 1 \cdot 6r + 1 \cdot 7r + 1} + \&c.$$

where the denominators are the same as in the given series, and the numerators in arithmetic progression; assume therefore

$$2cr - r + 1 \cdot b = m, \quad 2cr + r - 1 \cdot b = m + n, \quad \text{hence } b = \frac{n}{2r},$$

$c = \frac{2rm + r + 1 \cdot n}{4r^2}$, which, substituted in cor. 2. LEM. 2. give

$$\begin{aligned} &\frac{m}{r + 1 \cdot 2r + 1 \cdot 3r + 1} + \frac{m+n}{3r + 1 \cdot 4r + 1 \cdot 5r + 1} + \frac{m+3n}{5r + 1 \cdot 6r + 1 \cdot 7r + 1} + \&c. \dots \\ &= \frac{2r + 1 \cdot n - 2rm}{2r^3} \times S + \frac{2rm - 3r + 1 \cdot n}{4r^3} + \frac{2rm - r - 1 \cdot n}{4r^2 \cdot r + 1}. \end{aligned}$$

Cor. 1. In prop. 1. substitute a for m , and $2b$ for n , and we have

$$\frac{a}{1 \cdot r+1 \cdot 2r+1} + \frac{a+2b}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+4b}{4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots =$$

$$\frac{ra-r+1 \cdot b}{r^3} \times S + \frac{2r+1 \cdot b-ra}{2r^3}.$$

Also in this prop. substitute $a+b$ for m , and $2b$ for n , and we have

$$\frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1} + \frac{a+3b}{3r+1 \cdot 4r+1 \cdot 5r+1} + \&c. \dots =$$

$$\frac{r+1 \cdot b-ra}{r^3} \times S + \frac{ra-2r+1 \cdot b}{2r^3} + \frac{ra+b}{2r^2 \times r+1}.$$

Subtract this latter series from the former, and

$$\frac{a}{1 \cdot r+1 \cdot 2r+1} - \frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1} + \frac{a+2b}{2r+1 \cdot 3r+1 \cdot 4r+1} - \&c. \dots =$$

$$\frac{2ra-r+1 \cdot 2b}{r^3} \times S + \frac{2r+1 \cdot b-ra}{r^3} - \frac{ra+2b}{2r^2 \times r+1}.$$

Let $r=1$, and we have

$$\frac{a}{1 \cdot 2 \cdot 3} - \frac{a+b}{2 \cdot 3 \cdot 4} + \frac{a+2b}{3 \cdot 4 \cdot 5} - \&c. \dots = 2a - 4b \times S + \frac{11b-5a}{4}.$$

$$\text{If } a=1, b=0, \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} - \&c. \dots = 2S - \frac{5}{4};$$

$$a=1, b=2, \frac{1}{1 \cdot 2 \cdot 3} - \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} - \&c. \dots = \frac{17}{4} - 6S.$$

Let $r=3$, and we have

$$\frac{a}{1 \cdot 4 \cdot 7} - \frac{a+b}{4 \cdot 7 \cdot 10} + \frac{a+2b}{7 \cdot 10 \cdot 13} - \&c. \dots = \frac{6a-8b}{27} \times S + \frac{53b-33a}{216}.$$

$$\text{If } a=1, b=0, \frac{1}{1 \cdot 4 \cdot 7} - \frac{1}{4 \cdot 7 \cdot 10} + \frac{1}{7 \cdot 10 \cdot 13} - \&c. \dots = \frac{2}{9} S - \frac{11}{72};$$

$$a=1, b=1, \frac{1}{1 \cdot 4 \cdot 7} - \frac{2}{4 \cdot 7 \cdot 10} + \frac{3}{7 \cdot 10 \cdot 13} - \&c. \dots = \frac{5}{54} - \frac{2}{27} S.$$

If, instead of substituting in prop. 1. $2b$ and a for n and m , we had substituted two other quantities, as $2r$ and s , and then proceeded as above, a series would have been formed, the numerators of whose alternate terms would have formed each a separate arithmetic progression.

If the latter series had been *added* to the former a series would have been formed whose terms would have been all positive; but as I purpose, in the second part of this paper, to give a general method of summing all such series, I shall not stop here to apply this method of investigation.

Cor. 2. In proposition 2. substitute *a* for *m*, and *2b* for *n*, and we shall have

$$\frac{a}{r+1 \cdot 2r+1 \cdot 3r+1} + \frac{a+2b}{3r+1 \cdot 4r+1 \cdot 5r+1} + \&c. \dots =$$

$$\cdot \frac{2r+1 \cdot b-ra}{r^3} \times S + \frac{ra-3r+1 \cdot b}{2r^3} + \frac{ra-r-1 \cdot b}{2r^2 \cdot r+1}.$$

Also in prop. 1. write *a - b* for *m*, and *2b* for *n*, and there results

$$\frac{a+b}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+3b}{4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots =$$

$$\frac{ra-2r+1 \cdot b}{r^3} \times S + \frac{r+1 \cdot b-ra}{2r^3} - \frac{a-b}{r+1 \cdot 2r+1}.$$

Subtract this latter series from the former, and we shall have

$$\frac{a}{r+1 \cdot 2r+1 \cdot 3r+1} - \frac{a+b}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+2b}{3r+1 \cdot 4r+1 \cdot 5r+1} - \&c. \dots$$

$$= \frac{2r+1 \cdot 2b-2ra}{r^3} \times S + \frac{ra-3r+1 \cdot b}{r^3} + \frac{ra-r-1 \cdot b}{2r^2 \cdot r+1} + \frac{a-b}{r+1 \cdot 2r+1}.$$

P R O P. III.

To find the sum of the infinite series $\frac{m}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} +$
 $\frac{m+n}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1} + \frac{m+2n}{4r+1 \cdot 5r+1 \cdot 6r+1 \cdot 7r+1} + \&c.$

This series resolves itself into

$$\frac{a}{1 \cdot r+1 \cdot 2r+1} - \frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1} + \frac{a+2b}{2r+1 \cdot 3r+1 \cdot 4r+1} - \&c.$$

for by reduction it becomes

F f f 2

3ra - b

$$\frac{3^r a - b}{1 \cdot r + 1 \cdot 2r + 1 \cdot 3r + 1} + \frac{3ar + 4r - 1 \cdot b}{2r + 1 \cdot 3r + 1 \cdot 4r + 1 \cdot 5r + 1} +$$

$$\frac{3ar + 8r - 1 \cdot b}{4r + 1 \cdot 5r + 1 \cdot 6r + 1 \cdot 7r + 1} + \&c.$$

where the numerators are in arithmetic progression, and the denominators the same as in the given series; assume therefore

$$3ra - b = m, 3ra + 4r - 1 \cdot b = m + n, \text{ hence } b = \frac{n}{4^r}, a = \frac{4^r m + n}{12r^2};$$

substitute these values into cor. 1. prop. 2. and we have

$$\frac{m}{1 \cdot r + 1 \cdot 2 + 1 \cdot 3r + 1} + \frac{m + n}{2r + 1 \cdot 3r + 1 \cdot 4r + 1 \cdot 5r + 1} +$$

$$\frac{m + 2n}{4r + 1 \cdot 5r + 1 \cdot 6r + 1 \cdot 7r + 1} + \&c. \dots = \frac{4^r m - 3^r + 2 \cdot n}{6r^4} \times S +$$

$$\frac{3r + 1 \cdot n - 2rm}{6r^4} - \frac{rm + n}{6r^3 \cdot r + 1}.$$

Let $r = 1$, and we have

$$\frac{m}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{m + n}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{m + 2n}{5 \cdot 6 \cdot 7 \cdot 8} + \&c. \dots = \frac{4m - 5n}{6} \times S + \frac{7n - 5m}{12}.$$

$$\text{If } m = 1, n = 1, \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{5 \cdot 6 \cdot 7 \cdot 8} + \&c. \dots = \frac{2}{3} S - \frac{5}{12};$$

$$m = 1, n = 1, \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{3}{5 \cdot 6 \cdot 7 \cdot 8} + \&c. \dots = \frac{1}{6} - \frac{1}{6} S;$$

$$m = 7, n = 5, \frac{7}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{12}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{17}{5 \cdot 6 \cdot 7 \cdot 8} + \&c. \dots = \frac{1}{2} S;$$

Let $r = 2$, and we have

$$\frac{m}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{m + n}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{m + 2n}{9 \cdot 11 \cdot 13 \cdot 15} + \&c. \dots = \frac{m - n}{12} \times S - \frac{19n - 16m}{288}.$$

$$\text{If } m = 1, n = 3, \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{4}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{7}{9 \cdot 11 \cdot 13 \cdot 15} + \&c. \dots = \frac{41}{288} - \frac{1}{6} S;$$

$$m = 1, n = 0, \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{1}{9 \cdot 11 \cdot 13 \cdot 15} + \&c. \dots = \frac{1}{12} S - \frac{1}{18}.$$

$$m = 19, n = 16, \frac{19}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{35}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{51}{9 \cdot 11 \cdot 13 \cdot 15} + \&c. \dots = \frac{1}{4} S;$$

Cor. If $n : m$ as $4r : 3r + 2$, the sum of the series can be accurately had; let therefore $n = 4r$ and $m = 3r + 2$, and we shall have

$$3r + 2$$

$$\frac{3r+2}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} + \frac{7r+2}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1} + \&c. \dots = \frac{1}{2r \cdot r+1}$$

If $r = 1$, $\frac{5}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{9}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{13}{5 \cdot 6 \cdot 7 \cdot 9} + \&c. \dots = \frac{1}{4}^*$;

$r = 2$, $\frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{2}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{3}{9 \cdot 11 \cdot 13 \cdot 15} + \&c. \dots = \frac{1}{96}$;

$r = 6$, $\frac{5}{1 \cdot 7 \cdot 13 \cdot 19} + \frac{11}{13 \cdot 19 \cdot 25 \cdot 31} + \frac{17}{25 \cdot 31 \cdot 37 \cdot 43} + \&c. \dots = \frac{1}{336}$.

P R O P. IV.

To find the sum of the infinite series $\frac{m}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} +$

$$\frac{m+n}{3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \frac{m+2n}{5r+1 \cdot 6r+1 \cdot 7r+1 \cdot 8r+1} + \&c.$$

This series resolves itself into

$$\frac{a}{r+1 \cdot 2r+1 \cdot 3r+1} - \frac{a+b}{2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+2b}{3r+1 \cdot 4r+1 \cdot 5r+1} - \&c.$$

for by reduction it becomes $\frac{3ra - r + 1 \cdot b}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} +$

$$\frac{3ra + 3r - 1 \cdot b}{3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \frac{2ra + 7r - 1 \cdot b}{5r+1 \cdot 6r+1 \cdot 7r+1 \cdot 8r+1} + \&c.$$

where the denominators are the same as in the given series, and the numerators also in arithmetic progression; put therefore

$$3ra - r + 1 \cdot b = m, \quad 3ra + 3r - 1 \cdot b = m + n, \quad \text{hence } b = \frac{n}{4r},$$

$a = \frac{4rm + r + 1 \cdot n}{12r^2}$, which, substituted in cor. 2. prop. 2. give

$$\frac{m}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{m+n}{3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots = \frac{5r+2 \cdot n - 4rm}{6r^4} \times S + \frac{2rm - 4r + 1 \cdot n}{6r^4} + \frac{2rm - r - 2 \cdot n}{12 \cdot r^3 \cdot r+1} + \frac{4rm - 2r - 1 \cdot n}{12r^2 \cdot r+1 \cdot 2r+1}.$$

* Vide DE MOIVRE's Mif. Anal. pag. 134.

Cor. 1. In prop. 3. write a for m and $2b$ for n , and we have

$$\frac{a}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} + \frac{a+2b}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1} + \&c. \dots =$$

$$\frac{2ra-3r+2 \cdot b}{3r^4} \times S + \frac{3r+1 \cdot b-ra}{3r^4} - \frac{ra+2b}{6r^3 \cdot r+1}$$

Also in this prop. write $a+b$ for m , and $2b$ for n , and we have

$$\frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+3b}{3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots =$$

$$\frac{3r+2 \cdot b-2ra}{3r^4} \times S + \frac{ra-3r+1 \cdot b}{3r^4} + \frac{ra+2b}{6r^3 \cdot r+1} + \frac{2ra+b}{6r^2 \cdot r+1 \cdot 2r+1}$$

subtract this latter series from the former, and we have

$$\frac{a}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} - \frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \&c. \dots =$$

$$\frac{4ra-3r+2 \cdot 2b}{3r^4} \times S + \frac{3r+1 \cdot 2b-2ra}{3r^4} - \frac{ra+2b}{3r^3 \cdot r+1} - \frac{2ra+b}{6r^2 \cdot r+1 \cdot 2r+1}$$

Let $r=1$, and we have

$$\frac{a}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{a+b}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{a+2b}{3 \cdot 4 \cdot 5 \cdot 6} - \&c. \dots = \frac{4a-10b}{3} \times S + \frac{83b-32a}{36}$$

If $a=1, b=0$, $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} - \&c. \dots = \frac{4}{3}S - \frac{8}{9}$

$a=3, b=1$, $\frac{1}{1 \cdot 2 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 6} - \&c. \dots = \frac{2}{3}S - \frac{13}{36}$

Let $r=3$, and we have

$$\frac{a}{1 \cdot 4 \cdot 7 \cdot 10} - \frac{a+b}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{a+2b}{7 \cdot 10 \cdot 13 \cdot 16} - \&c. \dots = \frac{12a-22b}{243} \times S + \frac{1027b-156a}{3 \cdot 81 \cdot 56}$$

If $a=1, b=1$, $\frac{1}{1 \cdot 4 \cdot 7 \cdot 10} - \frac{2}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{3}{7 \cdot 10 \cdot 13 \cdot 16} - \&c. \dots = \frac{871}{3 \cdot 81 \cdot 56} - \frac{10}{3 \cdot 81} S$

$a=4, b=3$, $\frac{1}{1 \cdot 7 \cdot 10} - \frac{1}{4 \cdot 10 \cdot 13} + \frac{1}{7 \cdot 13 \cdot 16} - \&c. \dots = \frac{13}{72} - \frac{2}{27} S$

Cor. 2. If $a : b$ as $3r+2 : 2r$, the sum of the series can be accurately found; take $\therefore a = 3r+2$, and $b = 2r$, and we shall have

$$\frac{3r+2}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} - \frac{5r+2}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \&c. \dots = \frac{1}{r+1 \cdot 2r+1}$$

If

$$\begin{aligned} \text{If } r=1, & \frac{5}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{7}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{9}{3 \cdot 4 \cdot 5 \cdot 6} - \&c. \dots = \frac{1}{60}; \\ r=2, & \frac{2}{1 \cdot 3 \cdot 5 \cdot 7} - \frac{3}{3 \cdot 5 \cdot 7 \cdot 9} + \frac{4}{5 \cdot 7 \cdot 9 \cdot 11} - \&c. \dots = \frac{1}{15}; \\ r=3, & \frac{11}{1 \cdot 4 \cdot 7 \cdot 10} - \frac{17}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{23}{7 \cdot 10 \cdot 13 \cdot 16} - \&c. \dots = \frac{1}{28}. \end{aligned}$$

P R O P. V.

To find the sum of the infinite series $\frac{m}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{m+n}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \frac{m+2n}{4r+1 \cdot 5r+1 \cdot 6r+1 \cdot 7r+1 \cdot 8r+1} + \&c.$

This series resolves itself into $\frac{a}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1} - \frac{a+b}{r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{a+2b}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1} - \&c.$; for by reduction this series becomes $\frac{4ra-b}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{4ra+6r-1 \cdot b}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \frac{4ra+12r-1 \cdot b}{4r+1 \cdot 5r+1 \cdot 6r+1 \cdot 7r+1 \cdot 8r+1} + \&c.$

where the numerators are in arithmetic progression, and the denominators the same as in the given series; assume therefore $4ra-b=m$, $4ra+6r-1 \cdot b=m+n$, hence $b = \frac{n}{6r}$, $a = \frac{6rm+n}{24r^2}$, which values being substituted in cor. 1. prop. 4. give

$$\frac{m}{1 \cdot r+1 \cdot 2r+1 \cdot 3r+1 \cdot 4r+1} + \frac{m+n}{2r+1 \cdot 3r+1 \cdot 4r+1 \cdot 5r+1 \cdot 6r+1} + \&c. \dots = \frac{2rm-2r+1 \cdot n}{6r^3} \times S + \frac{4r+1 \cdot n-2rm}{12r^3} - \frac{6rm+9n}{72 \cdot r^4 \cdot r+1} - \frac{2rm+n}{24r^3 \cdot r+1 \cdot 2r+1}.$$

Let $r=1$, and we have

$$\frac{m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{m+n}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{m+2n}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c. \dots = \frac{2m-3n}{6} \times S + \frac{25n-16m}{72}.$$

$$\text{If } m=1, n=0, \frac{1}{1.2.3.4.5} + \frac{1}{3.4.5.6.7} + \frac{1}{5.6.7.8.9} + \&c. \dots = \frac{1}{3}S - \frac{2}{9};$$

$$m=1, n=1, \frac{1}{1.2.3.4.5} + \frac{2}{3.4.5.6.7} + \frac{3}{5.6.7.8.9} + \&c. \dots = \frac{1}{8} - \frac{1}{6}S;$$

$$m=4, n=2, \frac{1}{1.2.3.4.5} + \frac{1}{3.4.5.7} + \frac{1}{5.6.7.9} + \&c. \dots = \frac{1}{3}S - \frac{7}{36}.$$

$$m=25, n=16, \frac{25}{1.2.3.4.5} + \frac{41}{3.4.5.6.7} + \frac{57}{5.6.7.8.9} + \&c. \dots = \frac{1}{3}S.$$

Let $r=2$, and we have $\frac{m}{1.3.5.7.9} + \frac{m+n}{5.7.9.11.13} +$
 $\frac{m+2n}{9.11.13.15.17} + \&c. \dots = \frac{4^m - 5^n}{192} \times S + \frac{59^n - 44^n}{24.120}.$

$$\text{If } m=1, n=0, \frac{1}{1.3.5.7.9} + \frac{1}{5.7.9.11.13} + \frac{1}{9.11.13.15.17} + \&c. \dots = \frac{1}{48}S - \frac{11}{24.30};$$

$$m=3, n=2, \frac{1}{1.5.7.9} + \frac{1}{5.9.11.13} + \frac{1}{9.13.15.17} + \&c. \dots = \frac{1}{96}S - \frac{7}{24.60}.$$

Cor. If $n : m$ as $2r : 2r + 1$, the sum of the series can be accurately found; assume therefore $n = 2r, m = 2r + 1$, and we have

$$\frac{1}{1.r+1.3r+1.4r+1} + \frac{1}{2r+1.3r+1.5r+1.6r+1} + \&c. \dots = \frac{1}{6.r.r+1.2r+1}.$$

$$\text{If } r=1, \frac{1}{1.2.4.5} + \frac{1}{3.4.6.7} + \frac{1}{5.6.8.9} + \&c. \dots = \frac{1}{36};$$

$$r=3, \frac{1}{1.4.10.13} + \frac{1}{7.10.16.19} + \frac{1}{13.16.22.25} + \&c. \dots = \frac{1}{504}.$$

Having thus far explained the method of summation of such series as I proposed to treat of in the first part of this paper, I trust it is not necessary to say any thing further, as the same method of proceeding will manifestly continue the series to any proposed number of factors in the denominator; I shall therefore conclude with pointing out a remarkable property of those series whose sum can be accurately found: that when the number of factors in the denominator is *even*, the numerator is always equal to the sum of the two

middle factors; and when the number of factors be *odd*, the numerator will be equal to the middle factor, and consequently will take it out of the denominator, and leave a series whose numerators are unity, and whose denominators want the middle factor.

The method of summation of series here made use of may also be applied in investigating the sums of a great variety of other series; but as a further application of this method would carry us beyond the limits to which this paper must be confined, I shall re-assume the subject at some future opportunity, and proceed immediately to the second part.

P A R T II.

P R O P.

To find the sum of the infinite series $\frac{p}{n \cdot n+m \dots n+rm} + \frac{q}{n+m \dots n+r+1 \cdot m} + \frac{s}{n+2m \dots n+r+2 \cdot m} + \&c.$ *when the last differences of the numerators become equal to nothing.*

Assume $a + nb + n \cdot \overline{n+m} \cdot c + n \cdot \overline{n+m} \cdot \overline{n+2m} \cdot d + \&c.$ to any number (r) of terms; then, if for n we write $n+m, n+2m, n+3m, \&c.$ successively, there will result a series of quantities

CASE 2. To find the sum of the infinite series $\frac{1 \cdot 2}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{2 \cdot 3}{3 \cdot 5 \cdot 7 \cdot 9} + \frac{3 \cdot 4}{5 \cdot 7 \cdot 9 \cdot 11} + \&c.$

In this case $n = 1, m = 2, r = 3$, and the 3d differences become $= 0$; therefore $a + b + 3c = 2, a + 3b + 15c = 6, a + 5b + 35c = 12$, consequently $a = \frac{3}{4}, b = \frac{1}{2}, c = \frac{1}{4}$, and hence the sum of the required series becomes $\frac{3}{1 \cdot 3 \cdot 5 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 5 \cdot 2 \cdot 2 \cdot 2} + \frac{1}{5 \cdot 2 \cdot 4} = \frac{1}{24}$.

CASE 3. To find the sum of the infinite series $\frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{2}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{4}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \frac{12}{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{31}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} + \&c.$

Here $n = 3, m = 1, r = 5$, and the 4th differences become $= 0$; therefore $a + 3b + 12c + 60d = 1, a + 4b + 20c + 120d = 2, a + 5b + 30c + 210d = 4, a + 6b + 42c + 336d = 12$, consequently $a = -54, b = 47, c = -12, d = \frac{5}{6}$, therefore the sum of the given series is $\frac{-46}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{47}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} - \frac{12}{5 \cdot 6 \cdot 7 \cdot 8} + \frac{5}{6 \cdot 7 \cdot 8} = \frac{61}{50400}$.

CASE 4. To find the sum of the infinite series $\frac{1}{1 \cdot 4 \cdot 7 \cdot 10} + \frac{9}{4 \cdot 7 \cdot 10 \cdot 13} + \frac{25}{7 \cdot 10 \cdot 13 \cdot 16} + \&c.$

Here $n = 1, m = 3, r = 3$, and the 3d differences are $= 0$; therefore $a + b + 4c = 1, a + 4b + 28c = 9, a + 7b + 70c = 25$, consequently $a = \frac{1}{9}, b = \frac{-8}{9}, c = \frac{4}{9}$; \therefore the sum of the given series will be $\frac{1}{1 \cdot 4 \cdot 7 \cdot 3 \cdot 3 \cdot 9} - \frac{8}{4 \cdot 7 \cdot 3 \cdot 2 \cdot 9} + \frac{4}{7 \cdot 3 \cdot 9} = \frac{37}{2208}$.

CASE 5. To find the Sum of the infinite Series $\frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} +$
 $\frac{4}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} + \frac{10}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \frac{20}{7 \cdot 9 \cdot 11 \cdot 13 \cdot 15} +$

In this case $n = 1, m = 2, r = 4$, and the 4th differences become $= 0$; therefore $a + b + 3c + 15d = 1, a + 3b + 15c + 105d = 4, a + 5b + 35c + 315d = 10, a + 7b + 63c + 693d = 20$, consequently $a = \frac{5}{128}, b = \frac{3}{128}, c = \frac{1}{128}, d = \frac{1}{48}$, and hence the sum of the given series becomes $\frac{5}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 2 \cdot 4 \cdot 16} + \frac{3}{3 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 16} + \frac{1}{5 \cdot 7 \cdot 2 \cdot 2 \cdot 16} + \frac{1}{7 \cdot 2 \cdot 48} = \frac{1}{384}$.

This proposition may also be applied to find the sum of all those series whose numerators being unity, the denominators shall be deficient by any number of corresponding terms, however taken: for as the product of all such factors must form a progression, whose differences will become equal to nothing, if such products be assumed for the numerators of the given series having its factors completed, another series will be formed equal to the given series, whose sum can be found by this proposition.

CASE 1. To find the sum of the infinite series $\frac{1}{1 \cdot 2 \cdot 4 \cdot 6} +$
 $\frac{1}{2 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 6 \cdot 8} + \&c.$

By completing the factors in the denominators, and multiplying the numerators by the same quantities the given series becomes $\frac{15}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{24}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{35}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c.$ in which case $n = 1, m = 1, r = 5$, and the 3d differences become $= 0$;

= 0; therefore $a + b + 2c = 15$, $a + 2b + 6c = 24$, $a + 3b + 12c = 35$, consequently $a = 8$, $b = 5$, $c = 1$, and therefore the sum of the series required is

$$\frac{8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 3} = \frac{211}{7200}.$$

CASE 2. To find the sum of the infinite series $\frac{1}{1 \cdot 5 \cdot 11} +$

$$\frac{1}{3 \cdot 7 \cdot 13} + \frac{1}{5 \cdot 9 \cdot 15} + \&c.$$

This series, when completed, becomes $\frac{189}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} +$

$$\frac{495}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \frac{1001}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15} + \frac{1755}{7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17} + \&c.$$

where $n = 1$, $m = 2$, $r = 5$, and the 4th differences are = 0; therefore $a + b + 3c + 15d = 189$, $a + 3b + 15c + 105d = 495$, $a + 5b + 35c + 315d = 1001$, $a + 7b + 63c + 693d = 1755$, consequently $a = 96$, $b = 48$, $c = 10$, $d = 1$; and hence the sum of

$$\text{the given series is } \frac{96}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 2 \cdot 5} + \frac{48}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 2 \cdot 4} + \frac{10}{5 \cdot 7 \cdot 9 \cdot 2 \cdot 3} + \frac{1}{7 \cdot 9 \cdot 2 \cdot 2} = \frac{487}{18900}.$$

CASE 3. To find the sum of the infinite series $\frac{1}{1 \cdot 4 \cdot 13 \cdot 16} +$

$$\frac{1}{4 \cdot 7 \cdot 16 \cdot 19} + \frac{1}{7 \cdot 10 \cdot 19 \cdot 22} + \&c.$$

This series resolves itself into $\frac{70}{1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 16} +$

$$\frac{130}{4 \cdot 7 \cdot 10 \cdot 13 \cdot 16 \cdot 19} + \frac{208}{7 \cdot 10 \cdot 13 \cdot 16 \cdot 19 \cdot 22} + \&c.$$

where $n = 1$, $m = 3$, $r = 5$, and the 3d differences = 0; therefore $a + b + 4c = 70$, $a + 4b + 28c = 130$, $a + 7b + 70c = 208$, from whence $a = 54$, $b = 12$, $c = 1$; therefore the sum of the given series is

$$\frac{54}{1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 5} + \frac{12}{4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 4} + \frac{1}{7 \cdot 10 \cdot 13 \cdot 3 \cdot 3} = \frac{227}{53800}.$$

By this proposition we may also investigate the sum of the series when there are any number of deficient terms in the denominators, and where the last differences of the numerators become equal to nothing; for if the factors in the denominators be completed, and the numerators be multiplied by the same quantities, their differences will still become equal to nothing.

CASE 1. To find the sum of the infinite series $\frac{1}{1 \cdot 3 \cdot 4 \cdot 6} +$
 $\frac{3}{2 \cdot 4 \cdot 5 \cdot 7} + \frac{6}{3 \cdot 5 \cdot 6 \cdot 8} + \frac{10}{4 \cdot 6 \cdot 7 \cdot 9} + \frac{15}{5 \cdot 7 \cdot 8 \cdot 10} + \&c.$

This series, by completing the factors in the denominators and multiplying the numerators by the same quantities, becomes

$$\frac{10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{54}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{168}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c. \text{ in which}$$

case $n=1$, $m=1$, $r=5$, and as the 5th differences are $=0$;

$$\therefore a + b + 2c + 6d + 24e = 10, \quad a + 2b + 6c + 24d + 120e = 54,$$

$$a + 3b + 12c + 60d + 360e = 168, \quad a + 4b + 20c + 120d + 840e = 400,$$

$$a + 5b + 30c + 210d + 1680e = 810, \text{ from whence } a=0, \quad b=0,$$

$$c = -1, \quad d=0, \quad e = \frac{1}{2}, \text{ consequently the sum of the given series is}$$

$$= -\frac{1}{3 \cdot 4 \cdot 5 \cdot 3} + \frac{1}{5 \cdot 2} = \frac{17}{180}.$$

CASE 2. To find the sum of the infinite series $\frac{1}{1 \cdot 3 \cdot 7 \cdot 9} +$
 $\frac{5}{3 \cdot 5 \cdot 9 \cdot 11} + \frac{11}{5 \cdot 7 \cdot 11 \cdot 13} + \frac{19}{7 \cdot 9 \cdot 13 \cdot 15} + \&c.$

By proceeding as before this series becomes $\frac{3}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} +$

$$\frac{35}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} + \frac{99}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \&c.$$
 where $n=1, m=2, r=4$,
 and the 4th differences = 0; therefore $a+b+3c+15d=5$,
 $a+3b+15c+105d=35$, $a+5d+35c+315d=99$,
 $a+7b+63c+693d=209$, consequently $a=-1, b=\frac{3}{4}, c=\frac{1}{2}$,
 $d=\frac{1}{4}$, therefore the sum of the given series = $\frac{-1}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 2 \cdot 4} +$
 $\frac{3}{3 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \cdot 2} + \frac{1}{7 \cdot 2 \cdot 4} = \frac{3}{140}$.

By a method similar to that made use of in this proposition may any number of factors be taken from the denominators of those series delivered in part the first, and also from a great variety of others; but as the examples here given must be sufficient to point out the method of proceeding in all other cases, we will proceed to the third part.

P A R T III.

THE sum of every converging infinite series, whose terms ultimately become equal to nothing, may always be exhibited by the sum of another series formed by collecting two or more terms of the former series into one. This is not true, however, where the terms of the infinite series continually diverge, or converge to any assignable quantity, and

and are affected with the signs $+$, $-$, alternately: for instance, the series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c.$ if we collect two terms into one, beginning at the first term, will become $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} + \&c.$ If we begin at the second term it becomes $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c.$; neither of which gives the sum of the assumed series; but in this, and every other case of the like nature, a correction will be necessary: to determine the value of which, and from whence the necessity thereof arises, is the subject of this third part.

L E M M A.

If r be any quantity whatever, then will $\frac{1}{2r} = \frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \&c.$ ad infinitum.

For $\frac{1}{2r} = \frac{1}{r+r} =$ (by common division) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \&c.$ ad infinitum.

Cor. 1. Hence $-\frac{1}{2r} = -\frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \&c.$ ad infinitum.

Cor. 2. Hence also $\frac{x}{2v} = \frac{x}{v} - \frac{x}{v} + \frac{x}{v} - \frac{x}{v} + \&c.$ ad infinitum;

and $-\frac{x}{2v} = -\frac{x}{v} + \frac{x}{v} - \frac{x}{v} + \frac{x}{v} - \&c.$ ad infinitum.

P R O P. I.

If $\frac{rn+m}{n}$ be the general term of a series formed by writing for n any series of numbers in arithmetic progression, and whose signs are alternately $+$ and $-$; then if a series be formed by collecting

collecting two terms into one, beginning at the first term, the sum of the series thence arising will be less than the sum of the given series by $\frac{1}{2r}$. If a series be formed by beginning at the second Term, the sum thereof will be greater than the sum of the given series by $\frac{1}{2r}$.

For let $\frac{n}{rn+m} - \frac{n+a}{n+a \cdot r+m}$ be any two successive terms of the series, which, if we begin to collect at the first term (the first term being +) will be two terms to be collected into one, and which will therefore give $\frac{-am}{rn+m \times n+a \cdot r+m}$ for a general term of the resulting series. Let us now make n infinite, and then the denominator of this term becomes infinite, and the numerator finite; therefore the terms of this latter series at an infinite distance becoming infinitely small, the series will there terminate. Now, by making n infinite in the given series, the two successive general terms at an infinite distance become $\frac{1}{r} - \frac{1}{r}$; consequently *this* series is still continued after the *other* terminates; and the terms of such a continuation will be (as they begin with $\frac{n}{rn+m} - \frac{n+a}{n+a \cdot r+m}$ by making n infinite) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \&c.$ which will also be continued ad infin. and whose sum by the lemma is $\frac{1}{2r}$; consequently the given series exceeds that which is formed by collecting two terms into one, beginning at the first, by $\frac{1}{2r}$; hence the sum of the *latter* series + $\frac{1}{2r}$ will be equal to the sum of the *former*. If we begin to

collect at the second term, then will $-\frac{n}{rn+m} + \frac{n+a}{n+a.r+m}$ be the two successive general terms of the given series to be collected into one; consequently the continuation of the given series when n becomes infinite will be $-\frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \&c.$ ad infinitum whose sum, by cor. 1. to the lem. is $-\frac{1}{2r}$; in this case, therefore, the sum of the given series is less than the sum of the series formed by collecting two terms into one, beginning at the second term, by $\frac{1}{2r}$; hence the sum of the *latter* series $-\frac{1}{2r}$ will be equal to the sum of the *former*.

CASE 1. Let the given series be $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c.$

Here $r = 1$, $n = 1, 2, 3, 4, \&c.$ and $m = 1$. Now, if we begin to collect at the first term, the series resolves itself into $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} - \&c.$ and the correction, to be *added*, being $\frac{1}{2}$, we have $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} - \&c. + \frac{1}{2}$ for the sum of the given series. Now $-\frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} - \&c.$ is well known to be equal to $-1 + \text{hyp. log. of } 2$; consequently the sum of the given series is $= -\frac{1}{2} + \text{hyp. log. of } 2$.

If we begin to collect at the second term, the series becomes $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c.$ and the correction, to be *subtracted*, being $\frac{1}{2}$, we have $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c. - \frac{1}{2}$ for the sum of the given series;

series; but $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c.$ is equal to the hyp. log. of 2; therefore the sum of the given series is $= -\frac{1}{2} + \text{hyp. log. of } 2,$ the same as before.

CASE 2. Let the given series be $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \&c.$

Here $r=2, n=1, 2, 3, 4, \&c. m=1.$ Now, if we begin to collect at the second term, the series becomes $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \&c.$ and the correction, to be *subtracted*, being $\frac{1}{4}$, we have $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \&c. - \frac{1}{4}$ for the sum of the given series; but $\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \&c.$ is equal to a circular arc (A) of $22^{\circ}\frac{1}{2}$, whose radius is unity; therefore the sum of the given series $= A - \frac{1}{4}.$

P R O P. II.

If $\frac{x + nx}{w + nv}$ be the general term of a series formed by writing for n any series of numbers in arithmetic progression, and whose terms are alternately + and -; then if a series be formed by collecting two terms into one, beginning at the first term, the sum of the series thence arising will be less than the sum of the given series by $\frac{z}{2v}.$ If a series be formed by beginning at the second term, the sum thereof will be greater than the sum of the given series by $\frac{z}{2v}.$

For let $\frac{x+nz}{w+nv} - \frac{x+n+a \cdot z}{w+n+a \cdot v}$ be any two successive terms of the given series, which, if we begin to collect at the first term, will be the two general terms to be collected into one, and will therefore give $\frac{axv - awz}{w+nv \times w+n+a \cdot v}$ for a general term of the resulting series.

Let us now make n infinite and then this term will vanish, and consequently the resulting series will terminate at an infinite distance. Now, by making n infinite in the given series, the two successive terms (as they begin with $\frac{x+nz}{w+nv} - \frac{x+n+a \cdot z}{w+n+a \cdot v}$ by making n infinite) become $\frac{z}{v} - \frac{z}{v}$; this series, therefore, is still continued after the other terminates; and the terms of such a continuation will be $\frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \&c.$ ad infinitum, and whose sum by cor. 2. to the lem. is $\frac{z}{2v}$; consequently the given series exceeds that which is formed by collecting two terms into one, beginning at the first, by $\frac{z}{2v}$; hence the sum of the *latter* series $+ \frac{z}{2v}$ will be equal to the sum of the *former*. Now, if we begin to collect at the second term, then will $-\frac{x+nz}{w+nv} + \frac{x+n+a \cdot z}{w+n+a \cdot v}$ be two general terms of the given series to be collected into one; consequently the continuation of the given series, when n becomes infinite, will be $-\frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \&c.$ ad infinitum, whose sum by cor. 2. to the lem. is $-\frac{z}{2v}$; in this case, therefore, the sum of the given series is less than the sum of the series formed by collecting two terms into one, beginning at the second term, by

$\frac{x}{2v}$; hence the sum of the latter series $-\frac{x}{2v}$ will be equal to the sum of the former.

CASE 1. Let the given series be $\frac{7}{3} - \frac{11}{5} + \frac{15}{7} - \frac{19}{9} + \&c.$

Here $x = 3$, $\alpha = 2$, $w = 1$, $v = 1$, $n = 2, 4, 6, 8, \&c.$ Now, if we begin to collect at the first term, the series becomes

$\frac{2}{3 \cdot 5} + \frac{2}{7 \cdot 9} + \frac{2}{11 \cdot 13} + \&c.$ and the correction, to be added, being 1,

we have $\frac{2}{3 \cdot 5} + \frac{2}{7 \cdot 9} + \frac{2}{11 \cdot 13} + \&c. + 1$ for the sum of the given

series; but if $A =$ a circular arc of 45° whose radius is

unity, it is well known that $\frac{2}{3 \cdot 5} + \frac{2}{7 \cdot 9} + \frac{2}{11 \cdot 13} + \&c. = 1 - A$;

therefore the sum of the given series is $2 - A.$

CASE 2. Let the given series be $\frac{16}{1} - \frac{27}{2} + \frac{38}{3} - \frac{49}{4} + \&c.$

Here $w = 1$, $v = 1$, $x = 16$, $\alpha = 11$, $n = 0, 1, 2, 3, \&c.$ Now, if we begin to collect at the first term, the series becomes

$\frac{5}{1 \cdot 2} + \frac{5}{3 \cdot 4} + \frac{5}{5 \cdot 6} + \&c.$ and the correction, to be added, being

$\frac{11}{2}$, we have $\frac{5}{1 \cdot 2} + \frac{5}{3 \cdot 4} + \frac{5}{5 \cdot 6} + \&c. + \frac{11}{2}$ for the sum of the given

series; but $\frac{5}{1 \cdot 2} + \frac{5}{3 \cdot 4} + \frac{5}{5 \cdot 6} + \&c.$ is equal to $5 \times$ hyp. log. of 2,

consequently the sum of the given series is equal to

$\frac{11}{2} + 5 \times$ hyp. log. of 2.

Because $\frac{axv - avx}{w + nv \times w + n + a \cdot v}$, the general term of the series

formed

formed by reducing two terms into one, has its numerator independent of the value of, n , it is manifest, that the numerators of that series will be all equal. Now, if a series be assumed, the numerators of whose terms are unity, and in every other respect the same as the series in this proposition, that is, if

$\frac{1}{w+nv} - \frac{1}{w+n+a.v}$ be two successive terms of a series, it is manifest,

that if every two terms of this series be reduced into one, the general term of the resulting series will be

$\frac{-av}{w+nv \times w+n+a.v}$, where the numerator is a constant quantity

$-av$; consequently the sum of the series whose general term is

$\frac{avx-awz}{w+nv \times w+n+a.v}$ is to the sum of the series whose general term

is $\frac{-av}{w+nv \times w+n+a.v}$ as $vx-wz$ to $-v$, or in a given ratio;

whenever, therefore, the sum of the latter series can be found, the sum of the former can be found, and consequently, after proper correction, the sum of the series in this proposition can be found.

Hence, therefore, in the two cases given above in whatever arithmetic progression the numerators may proceed, the sum of the former can always be expressed by circular arcs, and the latter by the hyp. log. of 2.

Hence also, as it appears from lem. 1. part the first, that the sum of the series $\frac{1}{1} - \frac{1}{r+1} + \frac{1}{2r+1} - \frac{1}{3r+1} + \&c.$ can always be expressed by circular arcs and logarithms, it is manifest, that if the numerators form any arithmetic progression, the sum of such series may be found by this proposition, and will always be exhibited by circular arcs and logarithms.

Besides

Besides the series contained in the foregoing propositions, a great variety of other series might be produced where a correction is necessary, after collecting two terms into one, in order to exhibit the true value of the given series. As the proper correction, however, may always be found from the principles delivered in the above propositions, that is, by considering what the terms of the given series become at an infinite distance, I shall only add one or two instances more, and conclude what I at present intend to offer on this subject.

EX. 1. *Let it be required to find the sum of the infinite series*

$$\frac{3 \cdot 4}{1 \cdot 2} - \frac{4 \cdot 5}{2 \cdot 3} + \frac{5 \cdot 6}{3 \cdot 4} - \frac{6 \cdot 7}{4 \cdot 5} + \&c.$$

This, by resolving two terms into one, becomes $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} - \&c.$; and as the terms of the given series continually approach to unity, the correction, to be *added*, is $\frac{1}{2}$, consequently $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} - \&c. + \frac{1}{2}$ is equal to the sum of the given series; but by prop. 1. part I. the sum of the series $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} + \&c.$ is equal to $8S - 2$ (S being the hyp. log. 2.) consequently the sum of the given series is $8S - 1\frac{1}{2}$.

EX. 2. *Let it be required to find the sum of the infinite series*

$$\frac{1 \cdot 2}{1 \cdot 3} - \frac{2 \cdot 3}{3 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 7} - \frac{4 \cdot 5}{7 \cdot 9} + \&c.$$

This series, by resolving two terms into one, becomes $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \&c.$ and as the terms of the given series

series continually approach to $\frac{1}{4}$, the correction, to be added, will be $\frac{1}{8}$, therefore $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \&c. + \frac{1}{8}$ is = to the sum of the given series; but by prop. 1. part I. the sum of $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \&c.$ is equal to $\frac{1}{4}S + \frac{1}{8}$ (S being a circular arc of 45° , whose radius is unity) hence the sum of the given series is $\frac{1}{4}S + \frac{1}{4}$.

This method is not only applicable to those cases, where the given series resolves itself into another, whose sum is either accurately known or can be expressed by circular arcs and logarithms, but also to those cases where we want to approximate to the value of the given series, as it must, in general, be necessary first to render the terms of the series converging, by collecting two into one, before the operation of approximation begins, and consequently a correction of this latter is necessary in order to exhibit the value of the given series.



E R R A T A.

Page. Line.

72. 2. *for* comunicated *read* communicated

77. 12. *for* mercury removed without causing this to rise *read* remove
without causing the mercury to rise

81. 9. *for* finaller towards the middle *read* larger towards the middle

391. — the last line, *for* — *read* +

403. 12. *for* — 54 *read* — 46

408. 3. from the bottom, *for* $\frac{rn+m}{n}$ *read* $\frac{n}{rn+m}$